Student Workbook

Rotating Coordinate Axes

Name: ____________________ Date: ____________ Class: ________

ESSENTIAL QUESTION

How does rotating the coordinate axes eliminate the xy-term from a conic equation, and how do we use the discriminant B^2-4AC to identify the type of conic without rotating?

Lesson Overview

The general second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ represents a conic that may be tilted relative to the axes whenever $B \neq 0$. Rotating the axes by the right angle θ eliminates the xy-term, and the discriminant B^2-4AC — which never changes under rotation — instantly tells you the conic's type without doing any rotation at all.

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$
$$\cot(2\theta) = (A-C)/B$$

Rotation Formulas	$x = x'\cos\theta - y'\sin\theta$; $y = x'\sin\theta + y'\cos\theta$. Substituting these into the general equation and choosing θ so $\cot(2\theta)=(A-C)/B$ removes the x'y' term.
Discriminant Test	$B^2-4AC < 0$ → Ellipse (circle if A=C, B=0). = 0 → Parabola. > 0 → Hyperbola.
Invariants	$A + C$ and $B^2 - 4AC$ never change under rotation — use them to check your work.
θ = 45° Case	When A = C, $\cot(2\theta) = 0$, so θ = 45° always. $\cos 45° = \sin 45° = 1/\sqrt{2}$.

Worked Examples

EXAMPLE 1

Use the discriminant to identify: $2x^2 + 4xy + 3y^2 - 5x + 2y - 1 = 0$.

- A = 2, B = 4, C = 3
- $B^2-4AC = 16 - 4(2)(3) = 16 - 24 = -8$

Answer: Ellipse (discriminant = −8 < 0)

EXAMPLE 2

Find the angle of rotation to eliminate the xy-term in xy = 1.

- Rewrite: $0\cdot x^2 + 1\cdot xy + 0\cdot y^2 - 1 = 0$; A = 0, B = 1, C = 0
- $\cot(2\theta) = (A-C)/B = (0-0)/1 = 0$ → 2θ = 90° → θ = 45°

Answer: θ = 45°

EXAMPLE 3

Rotate $xy = 1$ by 45° and write in standard form.

- $x = x'\cos 45° - y'\sin 45° = (x' - y')/\sqrt{2}$; $y = x'\sin 45° + y'\cos 45° = (x' + y')/\sqrt{2}$
- $xy = (x' - y')(x' + y')/2 = (x'^2 - y'^2)/2 = 1$

Answer: $x'^2/2 - y'^2/2 = 1$ (hyperbola with $a = b = \sqrt{2}$)

EXAMPLE 4

Identify the conic: $5x^2 - 6xy + 5y^2 - 8 = 0$.

- $A = 5$, $B = -6$, $C = 5$; $B^2 - 4AC = 36 - 4(5)(5) = 36 - 100 = -64$
- $B^2 - 4AC < 0 \rightarrow$ Ellipse. $\cot(2\theta) = (5-5)/(-6) = 0 \rightarrow \theta = 45°$

Answer: Ellipse; rotate by 45°

EXAMPLE 5

Use the discriminant to identify: $x^2 + 4xy + 4y^2 + 2x - y = 0$.

- $A = 1$, $B = 4$, $C = 4$
- $B^2 - 4AC = 16 - 4(1)(4) = 16 - 16 = 0$

Answer: Parabola (discriminant = 0)

Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

1 Use the discriminant to identify: $3x^2 - 2xy + y^2 + x - 4 = 0$.

Hint: Compute $B^2 - 4AC$ with $A = 3$, $B = -2$, $C = 1$.

2 Find the angle of rotation for: $x^2 + 2\sqrt{3} \cdot xy - y^2 = 4$.

Hint: $\cot(2\theta) = (A-C)/B = (1-(-1))/(2\sqrt{3}) = 2/(2\sqrt{3}) = 1/\sqrt{3}$. What angle has $\cot = 1/\sqrt{3}$?

3 Identify: $4x^2 + 4xy + y^2 - 8x + 4y = 0$.

Hint: $B^2 - 4AC = 16 - 4(4)(1) = 0$. What type of conic has discriminant = 0?

4 For the rotation $x = x'\cos\theta - y'\sin\theta$, $y = x'\sin\theta + y'\cos\theta$ with $\theta = 30°$, express x and y in terms of x' and y'.

Hint: $\cos 30° = \sqrt{3}/2$, $\sin 30° = 1/2$.

5 Verify that B^2-4AC is invariant: show that for $2x^2+4xy+3y^2=1$, after rotating by the appropriate angle, the new equation has the same discriminant value.

Hint: After rotation, $B'=0$. New discriminant = $0^2-4A'C' = -4A'C'$. Use $A+C = A'+C'$ and $B^2-4AC = B'^2-4A'C'$.

Key Vocabulary

General 2nd-degree eq.	$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$; the xy-term indicates the conic is rotated.
Rotation of axes	Transforming (x,y) to (x',y') by rotating the coordinate system by angle θ.
Angle of rotation	θ chosen so that $\cot(2\theta) = (A-C)/B$, eliminating the x'y' term.
Discriminant	B^2-4AC; determines conic type without rotating.
Invariant	A quantity unchanged by rotation; $A+C$ and B^2-4AC are invariants.
Degenerate conic	A conic that reduces to a point, line, or pair of lines.

Quick Check Quiz

Circle the letter of the correct answer for each question.

1 **For $3x^2 + 5xy - 2y^2 + 1 = 0$, the discriminant B^2-4AC =** Multiple Choice

A $-24+25=1$
B $25+24=49$
C $25-24=1$
D -49

2 **$B^2-4AC = 0$ indicates a:** Multiple Choice

A Circle
B Ellipse
C Parabola
D Hyperbola

3 **The angle of rotation to eliminate xy in $x^2+2xy+y^2=1$ is:** Multiple Choice

A 30°
B 45°
C 60°
D 90°

4 **After rotating by 45°, xy=4 becomes:** Multiple Choice

A $x'^2-y'^2=8$
B $x'^2/8-y'^2/8=1$
C $x'^2-y'^2=4$
D $x'^2+y'^2=8$

5 **Which quantity is NOT invariant under rotation of axes?** Multiple Choice

A $A+C$

B B^2-4AC

C The value of B

D The type of conic

Common Mistakes to Avoid

✗ Forgetting that B is the coefficient of xy, not x^2y or xy^2.

✓ In $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, B is specifically the coefficient of the xy cross-term. Identify it carefully.

✗ Using $\cot(2\theta) = B/(A-C)$ instead of $(A-C)/B$.

✓ The correct formula is $\cot(2\theta) = (A-C)/B$. The numerator is $A-C$, denominator is B.

✗ Applying the discriminant test to equations not in general form.

✓ First write the equation as $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. Identify A, B, C correctly before computing B^2-4AC.

✗ Thinking the discriminant identifies the conic's orientation.

✓ The discriminant only identifies the TYPE (ellipse/parabola/hyperbola). It doesn't tell you the orientation or position.

Math Tips

★ Discriminant shortcut: $B^2-4AC < 0 \rightarrow$ Ellipse, $= 0 \rightarrow$ Parabola, $> 0 \rightarrow$ Hyperbola. Memorize: 'Less is Ellipse, Zero is Parabola, Greater is Hyperbola.'

★ When $B=0$, no rotation is needed — the conic is already aligned with the axes.

★ For $\theta=45°$: $\cos 45°=\sin 45°=1/\sqrt{2}$. The rotation formulas simplify to $x=(x'-y')/\sqrt{2}$, $y=(x'+y')/\sqrt{2}$.

★ After rotation, verify $B'=0$ in the new equation. If not, you made an arithmetic error.

★ The sum $A+C$ is invariant: after rotation, $A'+C' = A+C$. Use this as a check.